Multiple Perron-Frobenius operators

Yu. Dabaghian*
Department of Physics, Wesleyan University, Middletown, Connecticut 06459-0155
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A cycle expansion technique for discrete sums of several PF operators, similar to the one used in the standard classical dynamical ζ-function formalism is constructed. It is shown that the corresponding expansion coefficients show an interesting universal behavior, which illustrates the details of the interference between the particular mappings entering the sum.

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I. INTRODUCTION

Using the formalism of dynamical ζ functions, one can compute global averages of the observables, associated with a fully chaotic dynamical system,

$$\dot{x}_i = F_i(x). \tag{1}$$

In order to find the average for an observable $\Phi(x)$, one defines a Perron-Frobenius operator,

$$L_\Phi(x,y) = \delta[y_\mu - f_\mu(x)]e^{\beta\Phi(x)}, \tag{2}$$

where $f_\mu(x)$ is a flow associated with the system (1), and β is a parameter. The time average of $\Phi(x)$ can be found as

$$\langle\Phi\rangle = \frac{\partial}{\partial\beta}\ln z_{\min}(\beta)|_{\beta=0},$$

where $z_{\min}(\beta)$ is the smallest root of the eigenvalue equation

$$\det(1-zL) = 0. \tag{3}$$

For fully chaotic (axiom A) systems, the determinant (3) is an entire analytic function of z, which can be written [1–3] in an infinite product form as

$$\det(1-zL) = \prod_m \prod_p \left(1 - \frac{z^{n_p} e^{\beta\Phi(x)n_p}}{|\Lambda_p|\Lambda_p^m}\right)$$

$$\equiv \prod_m \prod_p (1 - z^{n_p} A_p^{n_p} t_p). \tag{4}$$

Here

$$t_p \equiv \frac{1}{|\Lambda_p|\Lambda_p^m}$$

is the weight of the prime periodic orbit of the system (1) indexed by p, Λ_p is it's Lyapunov exponent, A_p is a short notation for the amplitude, and m enumerates repetitions of periodic orbits. The product over the prime periodic orbits only,

$$\zeta_m^{-1}(z) = \prod_p (1 - z^{n_p} A^{n_p} t_p),$$

is known [2] as the dynamical ζ function associated with the system (1), which is a meromorphic function in a certain domain of the complex variable z [1–3].

An important feature of the theory based on the dynamical ζ function considerations, is the possibility to determine the coefficients of the power expansion of this function,

$$\zeta_m^{-1} = 1 - t_f z - c_2 z^2 - c_3 z^3 - \cdots,$$

using the very effective and insightful ''cycle expansion'' technique [4]. As an example, in the case of a system whose orbits can be described by a simple binary code, one can write the following cycle expansion of the ζ function ζ_m^{-1}:

$$\zeta_m^{-1} = (1-zt_0)(1-zt_1)(1-z^2t_{01})(1-z^3t_{001})(1-z^3t_{011})\cdots$$
$$= 1 - (t_1+t_0)z - (t_{01}-t_0t_1)z^2 - [(t_{001}-t_0t_{01}) + (t_{011}-t_1t_{01})]z^3 + \cdots.$$

It is easy to observe that in the ''curvature coefficients,'' which are in this case,

$$c_2 = t_{01} - t_0t_1,$$
$$c_3 = (t_{001}-t_0t_{01}) + (t_{011}-t_1t_{01}), \tag{5}$$
$$\cdots,$$

the contribution from long orbits is mimicked by the combinations of the short ones, which contribute to the sum with the opposite sign. As a result, the magnitude of the coefficients c_i of this expansion rapidly decrease. This allows to estimate very effectively the asymptotics of the coefficients of the expansion [4,1], and to prove the analyticity properties stated above.

II. GENERALIZATIONS

The basis of the PF operators theory was developed by Grothendieck in Ref. [5]. His ideas were generalized recently by Ruelle [6] and Kitaev [7], who proposed considering formal sums or integrals of PF operators L_ω, depending on a certain (discrete or continuous) parameter ω:

*Email address: ydabaghian@wesleyan.edu

$$\mathbf{L}(x,y)=\int \mu(d\omega)L_{\omega}[x-f_{\omega}(y)]. \tag{6}$$

Here, $\mu(d\omega)$ is some appropriate measure [6]. One of the most important requirements imposed on such a sum or an integral, is that the corresponding dynamical systems have to be fully chaotic for all the values of $\omega\in\Omega$.

As shown in [6], under certain natural requirements imposed on the dynamical mapping function $f(x)$ and the amplitude, the corresponding ζ functions and the Fredholm determinants are analytic functions in a certain domain of the complex plane. However, the size of this domain is smaller than the one in the case of a single PF operator. Even if the individual systems $L_{\omega}(f_{\omega})$ have an entire Fredholm determinant Z and the ζ function, the size of the analyticity domain of the averaged operator $\langle L\rangle$ can be finite.

Physically, operators like (5) can be used to study chaotic systems influenced by noise. Also, as shown recently in [8], the quantum mechanical Green's function of certain vector quantum particles can be presented as a sum similar to (6). A recent series of publications offered some effective methods for dealing with a system perturbed by Gaussian noise, including that of smooth conjugations [9] and local matrix representation [10]. However within these techniques the connection with the cycle expansion is not transparent. In this paper we attempt to construct a cycle expansion technique for discrete versions of such sums, for which an analog with cycle expansion technique is easy to establish.

III. DOUBLE PF OPERATOR

The simplest ''generalized PF operator'' is a formal sum of two (noncommuting) PF operators (2),

$$L^{(2)}(x,y)=A_1\delta(y_{\mu}-f^t_{1\mu}(x))+A_2\delta(y_{\mu}-f^t_{2\mu}(x)). \tag{7}$$

Here, A_1 and A_2 are certain multiplicative amplitudes.

Following the standard procedure [1,2], in order to describe the asymptotics of the evolution determined by the operator $L^{(2)}(x,y)$, we need to evaluate its largest eigenvalue, or the smallest root of the equation

$$\det(1-zL^{(2)})=\exp-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{Tr}(L^{(2)})^n$$

$$=\exp\sum_{n=1}^{\infty}\sum_{|i|=n}\mathrm{Tr}(L_{i_1}\cdots L_{i_n})=0,$$

where $i_k=1,2$. If a sequence $L_{i_1}\cdots L_{i_n}$ $(i_k=1,2)$ is periodic, it can be written as a power of a shorter aperiodic string,

$$L_{i_1}\cdots L_{i_n}=(L_{i'_1}\cdots L_{i'_s})^q,$$

where $n=s\cdot q$. Also, since only the trace of these operators is considered, the sequences obtained from one another by cyclic permutations give identical contributions, and therefore any length-s string contributes exactly s times. Hence,

$$\sum_{|i|=n}\mathrm{Tr}(L_{i_1}\cdots L_{i_n})=\sum_{q=1}\sum_{|i|=s}s\cdot\mathrm{Tr}(L_{i'_1}\cdots L_{i'_s})^q,$$

$$i_1,\ldots,i_n=1,2,$$

and for the spectral determinant (7) one has

$$\ln\det(1-zL^{(2)})=\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\sum_{|i|=s}\frac{z^{s\cdot q}}{q}\mathrm{Tr}(L'_{i_1}\cdots L_{i'_s})^q,$$

where the sum is written in terms of the powers q of the *aperiodic* strings of the length s of the symbols L_1 and L_2.

Using the formula

$$\delta[y_{\mu}-f_{1,\mu}(z_1)]\cdots\delta[z_{n,\mu}-f_{2,\mu}(x)]=\delta[y_{\mu}-f_{1,\mu}\circ f_{2,\mu}\circ\cdots\circ f_{n,\mu}(x)],$$

one gets the following expression for the trace of a generic term in the exponent:

$$\mathrm{Tr}(L_{i_1}\cdots L_{i_s})^q=\int L_{i_1}(x,y_1)\cdots L_{i_n}(y_{n-1},x)dx\,dy_1\cdots dy_{n-1}=\int\delta(x-f^t_{i_1}(y))\cdots\times\delta(y-f^t_{i_n}(x))dx\,dy_1\cdots dy_{n-1}$$

$$=\int\delta[x-f_{i_1}(y)\circ\cdots\circ f_{i_n}(x)]=\sum_w\frac{1}{|1-\Lambda^{(s)}_{w,q}|},$$

where w's are the fixed points of the map $f_{i_1}(y_1)\circ\cdots\circ f_{i_n}(x)$. The numbers s_1 and s_2 show how many times the operators L_1 and L_2 appear in the string $L_{i_1}\cdots L_{i_s}$. Obviously, $s_1+s_2=s$. Here, for simplicity, we put $A_1=A_2=1$, to avoid unnecessary complications in formulas. The dependence on the amplitudes is easy to reconstruct at the final step, so they will be ignored until the very last section.

Hence, there is an infinite number of nonequivalent operators $L_{i'_1}\cdots L_{i'_s}$ corresponding to each aperiodic string, and we are looking at their fixed points. Proceeding as usual, we can expand the denominator:

$$\frac{1}{|1-\Lambda^{(s)}_{w,q}|}=\sum_w\sum_{m=0}^{\infty}\frac{1}{|\Lambda^{(s)}_{w,q}|\Lambda^{(s)m}_{w,q}}.$$

Here, (s) denotes any of the aperiodic sequences of the length s, and $\Lambda_p^{(s)}$ represents the expansion factor of a prime orbit indexed by ''p'' of a composite operator $L_{i_1}\cdots L_{i_s}$. Substituting this expansion series into the exponent yields the following Fredholm determinant:

$$\det(1-zL)=\exp-\sum_{(s)}\sum_{p}\sum_{m=0}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{z^{s\cdot n_p}}{|\Lambda_p^{(s)}|\Lambda_p^{(s)m}}\right)^r=\prod_{m=0}^{\infty}\prod_{(s)}\prod_{p}\left(1-\frac{z^{s\cdot n_p}}{|\Lambda_p^{(s)}|\Lambda_p^{(s)m}}\right).$$

Correspondingly, the product

$$\zeta_m^{-1}=\prod_{(s)}\prod_{p}\left(1-\frac{z^{s\cdot n_p}}{|\Lambda_p^{(s)}|\Lambda_p^{(s)m}}\right) \tag{8}$$

can be considered as a generalization of the ζ function of the single-PF operator (3).

This object is quite different from it's one-mapping counterpart. While the single-mapping ζ function (3) is defined on the periodic orbits of the system (1), the product ζ_m^{-1} is taken over the fixed points of the (aperiodic) combinations $f_{i_1'}\circ\cdots\circ f_{i_s'}$, which can be thought of as ''interference'' terms. Expanding the (s) product in (8), one has

$$\zeta_m^{-1}=\prod_{p}\left(1-\frac{z^{n_p}}{|\Lambda_p^{(1)}|\Lambda_p^{(1)m}}\right)\left(1-\frac{(z^2)^{n_p}}{|\Lambda_p^{(2)}|\Lambda_p^{(2)m}}\right)\times\left(1-\frac{(z^3)^{n_p}}{|\Lambda_p^{(3)}|\Lambda_p^{(3)m}}\right)\cdots. \tag{9}$$

In order to avoid unnecessary complication of formulas, the repetition index m will be suppressed in the sequel. The parentheses (1), (2), (3) and so on imply the product over all the possible aperiodic combinations $L_{i_1}\cdots L_{i_s}$, $i=1,2$, of lengths 1, 2, 3 correspondingly. For instance, the first bracket consists of the simple product of two ζ functions (two possible length-one strings),

$$\prod_{p}(1-z^{n_p}t_p^{(1)})=\prod_{p}(1-z^{n_p}t_p^{1})\prod_{p}(1-z^{n_p}t_p^{2}),$$

where

$$t_p^i\equiv\frac{1}{|\Lambda_p^i|\Lambda_p^m}.$$

This is just the product of the individual ζ functions,

$$\zeta^{-1}(z|L_1)=\prod_{p}(1-z^{n_p}t_p^{1})=1-t_f^1z-c_2^1z^2-c_3^1z^3\ldots,$$

$$\zeta^{-1}(z|L_2)=\prod_{p}(1-z^{n_p}t_p^{2})=1-t_f^2z-c_2^2z^2-c_3^2z^3\ldots.$$

The term t_f^1 and t_f^1 represent the fundamental cycles, and the coefficients c_i^1 and c_i^2 denote the curvature corrections for the first and the second mapping correspondingly.

The only length-two string L_1L_2 (the first ''interference'' correction) produces it's own ζ function, which contributes the term

$$\zeta^{-1}(z|L_1L_2)=\prod_{p}(1-z^{2n_p}t_p^{12})=1-t_f^{12}z^2-c_2^{12}z^4-c_3^{12}z^6\ldots,$$

to the product. Here t_f^{12}, c_2^{12}, c_3^{12}, etc., are the fundamental cycles and the curvature corrections for the mapping $f_1\circ f_2(x)$.

The two length-three strings $L_1L_1L_2$ and $L_1L_2L_2$ contribute

$$\zeta^{-1}(z|L_1L_1L_2)=\prod_{p}(1-z^{3n_p}t_p^{112})=1-z^3t_f^{112}-c_2^{112}z^6-c_3^{112}z^9\ldots,$$

and

$$\zeta^{-1}(z|L_1L_2L_2)=\prod_{p}(1-z^{n_p}t_p^{122})=1-z^3t_f^{122}-c_2^{122}z^6-c_3^{122}z^9\ldots,$$

correspondingly, and so on. The top index shows the aperiodic strings of operators L_1 and L_2, while the lower index enumerates the prime orbits of each operator. $t_f^{(s)}$ represents the fundamental cycle of a given (s) operator, $L_{i_1}\cdots L_{i_s}$, and the coefficients $c_n^{(s)}$ give the corresponding curvature corrections.

It is important to notice that the symbolic dynamics of the individual mappings, used to obtain the expressions for $\zeta^{-1}(z|L_i)$, $\zeta^{-1}(z|L_1L_2)$, $\zeta^{-1}(z|L_1L_1L_2)$, $\zeta^{-1}(z|L_1L_2L_2)$, etc., do not need to be specified. In fact, one might use different symbolic dynamics in order to obtain the power expansion for these functions. After the necessary expansion series are at hand, the details of the symbolic dynamics of the corresponding mappings are irrelevant. The further analysis is made entirely in terms of the $t_f^{(s)}$'s and $c_n^{(s)}$'s.

Let us consider the power expansion of all the ζ-products up to the third power of z. The expansion for the two-PF operator will then be

$$\begin{aligned}\zeta_2^{-1}(z)&=[1-t_f^1 z-c_2^1 z^2-c_3^1 z^3-O(z^4)][1-t_f^2 z-c_2^2 z^2-c_3^2 z^3-O(z^4)]\\&\times[1-z^2 t_f^{12}-O(z^4)][1-z^3 t_f^{112}-O(z^6)][1-z^3 t_f^{122}-O(z^6)]\\&=1-(t_f^1+t_f^2)z-[(t_f^{12}-t_f^1 t_f^2)+c_2^1+c_2^2]z^2-[(t_f^{122}-t_f^2 t_f^{12})-(t_f^{112}-t_f^1 t_f^{12})\\&-(t_f^1+t_f^2)(c_2^1+c_2^2)+t_f^1 c_2^1+t_f^2 c_2^2+c_3^1+c_3^2]z^3+O(z^4),\end{aligned}\tag{10}$$

and so on, where t_f^i's and c_n^i are the fundamental contributions and the curvature corrections of the individual mappings.

Comparing the expressions (10) and (4), one can see that the structure of the coefficients t_f and c_n is repeated on a ''new level.'' The ''fundamental'' contribution is now

$$T_f^{(2)}=t_f^1+t_f^2,$$

and the following terms have a structure similar to the corresponding ''curvature corrections'' in the case of a system with complete binary dynamics (4). For example the term

$$C_2\equiv t_f^{12}-t_f^1 t_f^1,$$

is formally identical to the c_2 of (5), the term C_3 also contains terms

$$(t_f^{122}-t_f^2 t_f^{12})+(t_f^1 t_f^{12}-t_f^{112})$$

of manifestly ''curvature correction'' type.

However, starting from C_3 there start to appear some additional curvature combinations, such as

$$(t_f^1+t_f^2)(c_2^1+c_2^2)-t_f^1 c_2^1-t_f^2 c_2^2.$$

Also, every term contains the sum of all the standard curvature coefficients c_n of the corresponding degree, associated with the ''pure'' maps $f_1(x)$ and $f_2(x)$.

In short notations, which are going to be used below, the expansion (10) can be written as

$$\begin{aligned}\zeta_2^{-1}(z)=1-zT_f-z^2\left(C_2+\sum_{i=1}^{2}c_2^i\right)-z^3\left(C_3-\sum_{i=1}^{2}t_f^i\sum_{i=1}^{2}c_2^i\right.\\\left.+\sum_{i=1}^{2}t_f^i c_2^i+\sum_{i=1}^{2}c_3^i\right)+O(z^4).\end{aligned}\tag{11}$$

It should be emphasized that the coefficients T_f and C_n are principally different from their usual, single PF counterparts t_f and c_n. They are constructed using the contributions of *different* mappings, not only the $f_1(x)$ and $f_2(x)$ but also from all their aperiodic superpositions, and therefore produce some ''interference'' effect.

IV. TRIPLE PF OPERATOR

One can easily write down the explicit form of the power expansion for the triple PF operator

$$L^{(2)}(x,y)=A_1\delta(y_\mu-f_{1\mu}^t(x))+A_2\delta(y_\mu-f_{2\mu}^t(x))+A_3\delta(y_\mu-f_{3\mu}^t(x)).\tag{12}$$

As in the double PF operator case, the expansion is made up to the third order:

$$\begin{aligned}\zeta_3^{-1}(z)&=\prod_{i=1}^{3}[1-t_f^i z-c_2^i z^2-c_3^i z^3-O(z^4)]\prod_{ij}^{3}[1-z^2 t_f^{ij}-O(z^4)]\prod_{ijk}^{3}[1-z^3 t_f^{ijk}-O(z^6)]\\&=1-z(t_f^1+t_f^2+t_f^3)-z^2[(t_f^{23}-t_f^2 t_f^3)+(t_f^{13}-t_f^1 t_f^3)+(t_f^{12}-t_f^1 t_f^2)+c_2^1+c_2^2+c_2^3]+z^3[(t_f^{223}-t_f^2 t_f^{23})\\&+(t_f^{233}-t_f^3 t_f^{23})+(t_f^{112}-t_f^1 t_f^{12})+(t_f^{122}-t_f^2 t_f^{12})+(t_f^{113}-t_f^1 t_f^{13})+(t_f^{133}-t_f^3 t_f^{13})+t_f^{213}+t_f^{132}-t_f^3 t_f^{12}-t_f^2 t_f^{13}\\&-t_f^1 t_f^{23}+t_f^{123}+t_f^1 t_f^2 t_f^3-t_f^1(c_2^2+c_2^3)-t_f^2(c_2^1+c_2^3)-t_f^3(c_2^1+c_2^3)+(c_3^1+c_3^2+c_3^3)]+O(z^4),\end{aligned}\tag{13}$$

where t_f, c_n are the fundamental contributions and the curvature corrections of the corresponding mappings.

It is easy to observe that the above expansion is very similar to the one obtained for a system with a complete ternary symbolic dynamics, for which the expansion coefficients are

$$t_f^{(3)}=t_3+t_2+t_1,$$

$$c_2^{(3)}=(t_{13}-t_1 t_3)+(t_{23}-t_2 t_3)+(t_{12}-t_1 t_2),$$

$$\begin{aligned}c_3^{(3)}&=(t_{233}-t_3 t_{23})+(t_{223}-t_2 t_{23})+(t_{133}-t_3 t_{13})+(t_{113}-t_1 t_{13})+(t_{122}-t_2 t_{12})\\&+(t_{112}-t_1 t_{12})+t_{213}+t_{123}-t_3 t_{12}-t_2 t_{13}-t_1 t_{23}+t_1 t_2 t_3.\end{aligned}$$

Using the notations $T_f^{(3)}$, $C_2^{(3)}$, $C_3^{(3)}$ for the corresponding combinations of prime contributions in (14), we get for the triple operator ζ functions:

$$\zeta_3^{-1}(z)=1-zT_f^{(3)}-z^2\left(C_2^{(3)}+\sum_{i=1}^{3}c_2^i\right)-z^3\left(C_3^{(3)}-\sum_{i=1}^{3}t_f^i\sum_{i=1}^{3}c_2^i+\sum_{i=1}^{3}t_f^ic_2^i+\sum_{i=1}^{3}c_3^i\right)+O(z^4).$$

Comparing this to the expansion of the double PF operator,

$$\zeta_2^{-1}(z)=1-zT_f^{(2)}-z^2\left(C_2^{(2)}+\sum_{i=1}^{2}c_2^i\right)-z^3\left(C_3^{(2)}-\sum_{i=1}^{2}t_f^i\sum_{i=1}^{2}c_2^i+\sum_{i=1}^{2}t_f^ic_2^i+\sum_{i=1}^{2}c_3^i\right)+O(z^4),$$

it is easy to observe that the curvature coefficients have the same structure, depending trivially on the number of the PF operators in the sum elements. The main difference is contained in the terms $C_2^{(3)}$, $C_3^{(3)}$ as opposed to $C_2^{(2)}$, $C_3^{(2)}$, because they are constructed as the curvature terms of systems with binary and ternary symbolic dynamics correspondingly.

V. *N* SUMS AND THE CONTINUOUS LIMIT

Previous analysis can be conducted in the exact same way for any N-operator sum ($N\geqslant 2$),

$$L^{(N)}(x,y)=\sum_{i=1}^{N}\delta(y_\mu-f_{i,\mu}^t(x)). \tag{14}$$

For the Fredholm determinant $\det(1-zL)$ one has

$$\det(1-zL)=\exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{Tr}(L^n)\right)$$

$$=\prod_m\prod_s\prod_p\left(1-\frac{z^{s\cdot n_p}}{|\Lambda_p^{(s)}|\Lambda_p^{(s)m}}\right). \tag{15}$$

Based on the previous examples and using the induction method, one can prove that the power expansion of the ζ function of an N-term PF operator is

$$\zeta_N^{-1}(z)=1-zT_f^{(N)}-z^2\left(C_2^{(N)}+\sum_{i=1}^{N}c_2^i\right)$$

$$-z^3\left(C_3^{(N)}+\sum_{i=1}^{N}t_f^i\sum_{i=1}^{N}c_2^i-\sum_{i=1}^{N}t_f^ic_2^i+\sum_{i=1}^{N}c_3^i\right)$$

$$+O(z^4), \tag{16}$$

where

$$T_f^{(N)}=\sum_{i=1}^{N}t_f^i,$$

and $C_2^{(N)}$, $C_3^{(N)}$, etc., have the structure identical to the curvature correction coefficients of a system with a complete N-ary symbolic dynamics. The extra terms which appear in the expansion (16),

$$\tilde{c}_2^{(N)}\equiv\sum_{i=1}^{N}c_2^i,$$

$$\tilde{c}_3^{(N)}\equiv\sum_{i=1}^{N}t_f^i\sum_{i=1}^{N}c_2^i-\sum_{i=1}^{N}t_f^ic_2^i+\sum_{i=1}^{N}c_3^i, \tag{17}$$

$$\ldots,$$

can be called ''direct curvature contributions'' as opposed to the ''interference curvatures'' $C_n^{(N)}$.

The direct curvature terms (17) depend trivially on the number of PF operators included into the sum (15). Among the interference curvature terms, only the fundamental contribution term T_f is just the direct sum of the individual fundamental contributions.

It is important to emphasize, that the complete curvature coefficients have a *universal form*,

$$\tilde{C}_n^{(N)}=C_n^{(N)}+\tilde{c}_n^{(N)}, \tag{18}$$

which provides an algorithm for evaluating the ζ-function expansion coefficients of the multiple PF operators to a given degree n. Moreover, the direct curvature coefficients depend trivially on the order of the sum (15). This allows to make certain statements about the behavior of the *continuous* sums,

$$\mathbf{L}(x,y)=\int\mu(d\omega)A_\omega\delta(y_\mu-f_\mu^t(x,\omega)), \tag{19}$$

from the point of view of the standard cycle expansion technique. Here the amplitudes A_ω and the mapping functions $f_\mu^t(x,\omega)$ depend continuously on the parameter ω.

After some elementary considerations it is easy to reconstruct the explicit amplitude dependence in the expansions of the MPF operators' ζ functions,

$$\zeta_N^{-1}=\prod_{(s)}\prod_p\left(1-\frac{z^{s\cdot n_p}A^{(s)n_p}}{|\Lambda_p^{(s)}|\Lambda_p^{(s)m}}\right), \tag{20}$$

where $A^{(s)}\equiv A_1^{s_1}\cdots A_2^{s_N}$. The product over the (s) includes all the possible strings in which the symbols $L_1,\ldots,L_N$ appear $s_1,\ldots,s_2$ times correspondingly, for all $s_1+\cdots+s_N=s$. The expansion of (20) yields:

$$\zeta_N^{-1}=1-z\sum_{i=1}^{N}A_i t_f^i-z^2\left(A^{(2)}C_2+\sum_{i=1}^{N}A_i^2 c_2^i\right)$$
$$-z^3\left(A^{(3)}C_3+\sum_{i=1}^{N}A_i^3 c_3^i-\sum_{i=1}^{N}A_i t_f^i\right.$$
$$\left.\times\sum_{i=1}^{N}A_i^2 c_2^i+\sum_{i=1}^{N}A_i^3 t_f^i c_2^i\right)+O(z^4).$$

Without loss of generality, one can assume that there exists such a set of discrete parameters $\omega_1,\ldots,\omega_N$, that the curvature corrections of the individual maps can be written as

$$t_f^i\equiv t_f(\omega_i),$$
$$c_n^i\equiv c_n(\omega_i). \tag{21}$$

In case the amplitudes A_ω and the mapping functions $f_\mu^t(x,\omega)$ in (20) depend continuously on the parameter ω, the coefficients (19) also depend continuously on ω, at least for a certain range of parameters. Therefore, the fundamental contribution coefficient will produce in the limit $N\to\infty$:

$$T_f^{(N)}=\sum_{i=1}^{N}A_i t_f^i\to T_f=\int t_f(\omega)\mu(d\omega).$$

The direct curvature contributions in the expansion (17) also can be written in the integral form:

$$\tilde{c}_2^{(N)}\equiv\sum_{i=1}^{N}A_i^2 c_2^i\equiv\sum_{i=1}^{N}A_i^2 c_2(\omega_i)\to\tilde{c}_2\equiv\int c_2(\omega)\mu(d^2\omega),$$

$$\tilde{c}_3\equiv\int t_f(\omega)\mu(d\omega)\int c_2(\omega)\mu(d^2\omega)$$
$$-\int t_f c_2\mu(d^3\omega)+\int c_3(\omega)\mu(d^3\omega),$$

$$\ldots.$$

On the other hand, the interference curvature coefficients $C_n^{(N)}$ the limit $N\to\infty$ are considerably more complicated, since the elements contained in the higher order interference curvatures $C_n^{(N)}$, $n\geqslant 3$, are labeled by the aperiodic strings *of complexity N*.

However, the $N\to\infty$ limit for the interference coefficients exist [1,9,10], and therefore ζ function ζ^{-1} has the following expansion form:

$$\zeta_{N\to\infty}^{-1}=1-z\int t_f(\omega)\mu(d\omega)-z^2\left(C_2^N+\int c_2(\omega)\mu(d^2\omega)\right)$$
$$-z^3\left(C_3^N+\int t_f(\omega)\mu(d\omega)\int c_2(\omega)\mu(d^2\omega)\right.$$
$$\left.+\int t_f c_2(\omega)\mu(d^3\omega)-\int c_3(\omega)\mu(d^3\omega)\right)$$
$$+O(z^4).$$

This expansion is analogous to the cumulant expansion of $\det(1-zL)$ obtained in [10].

VI. CONCLUSION

From a practical as well as from a theoretical point of view, it is certainly more convenient to treat the continuous sum case using the methods developed in [9,10], because these methods effectively circumvent some convoluted calculations, which are necessary to evaluate the expansion coefficients (17). However, for discrete versions of such ''noisy'' PF operators, the multiple sums (14), these methods cannot be applied directly. Meanwhile, as was shown above, such systems allow a direct treatment which is analogous to the standard cycle expansion technique. The detailed form of the corresponding expansion coefficients (17) reveals a remarkable structural simplicity and shows certain a familiar features characteristic for the single-operator expansion coefficients.

In practice, even in the case of a single PF operator, one can usually obtain only the first several terms of the cycle expansion series. In the case of multiple PF operators, the explicit form of the generalized curvatures (17) conveniently illustrates the details of the interference between the different maps constituting a multiple PF operator, using the language of the periodic cycles arising from the maps themselves as well as their compositions. This provides a nontrivial extension of the periodic orbit theory to a more complicated dynamical evolution which involves a few ''evolution channels.''

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